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Electrohydrodynamic stability of a slightly viscous jet

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Many electro-spraying devices raise to a high electric potential a pendant drop of weakly conducting fluid, which may adopt a conical shape from whose apex a thin, charged jet is emitted. Such a jet eventually breaks up into fine droplets, but often displays surprising longevity. This paper examines the stability of an incompressible cylindrical jet carrying surface charge in a tangential electric field, allowing for the finite rate of charge relaxation. The viscosity is assumed to be small so that the shear resulting from the tangential surface stress can be large, even for relatively small fields. This shear can suppress surface tension instabilities, but if too large, it excites electrical ones. For imperfect conductors, surface charge is redistributed by the rapid fluid reaction to variations in tangential stress as well as by conduction. Phase differences between the effects due to the tangential field and the surface charge lead to charge 'over-relaxation' instabilities, but the maximum growth rate can still be lower than in the absence of electric effects.

1. Introduction

The behaviour of a drop of poorly conducting liquid in a strong electric field is a classical problem, originating in the study of thunderstorms. As the external field strength is slowly increased, the shape of the drop adjusts to the resultant surface stresses. An initially spherical drop becomes elongated, and eventually loses stability to one of two mechanisms. During a period of dynamic evolution, the drop may either break up into two smaller drops, or may emit a thin jet from a pointed region of its surface (Sherwood 1988).

This latter behaviour is closely related to the electrostatic spraying process, depicted in figure 1. A liquid is passed through an aperture which is raised to a high electric potential. For suitable parameter ranges a steady state is reached, known as 'the electrohydrodynamic cone–jet' in which the fluid surface becomes almost conical, with a thin jet emanating from its apex. This jet eventually becomes unstable, breaking up into droplets. The diameter of these droplets is much smaller than that of the aperture, providing a mechanism for the production of fine sprays. The process has many practical applications, for example in paint spraying and crop spraying.

In the absence of the jet, Taylor (1964) showed using a static local analysis that the cone adopts the 'Taylor angle', 49.3°. As the field strength is increased and the jet is formed, dynamic effects become important and cones of smaller angles are observed. Not all parameter ranges permit the formation of a stable cone–jet, although the reasons for this are not fully understood. There is a range of fluid conductivities for which it is generally agreed that the phenomenon occurs, but it does not always manifest itself in experiments with impure water, for which the conductivity is relatively high. The difficulties in producing aqueous cone–jets can, however, be

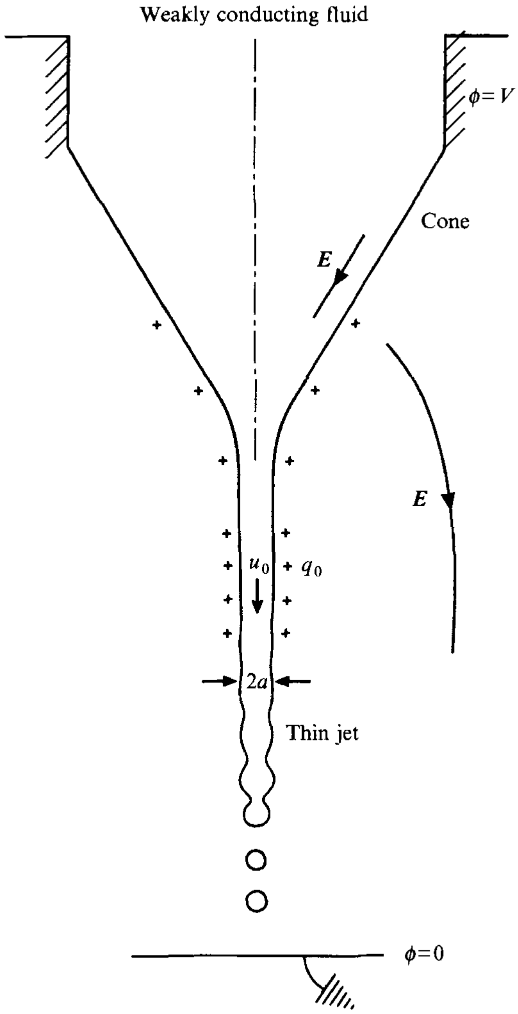

FIGURE 1. The electro-hydrodynamic cone–jet.

ascribed to the relatively high surface tension which necessitates an electric potential greater than the breakdown voltage of air (Zeleny 1915). Water cone–jets can be formed in other atmospheres, or if surfactants are added to lower the surface tension (Smith 1986). Drozin (1955) reported difficulty in producing jets at low permittivity, but these do occur if the conductivity is not too small (Jones & Thong 1971). Various other experimental results can be found in Zeleny (1917), Hayati, Bailey & Tadros (1987), Cloupeau & Prunet-Foch (1989), Bailey (1988), Michelson (1990), Fernández de la Mora *et al.* (1990), Fernández de la Mora & Loscertales (1993), and the references therein.

In this paper we concentrate on the question of the stability of the emitted jet. Previous studies have tended to consider either a charged jet in the absence of an external electric field (Bassett 1894; Taylor 1969; Saville 1971*b*) or an uncharged jet in a uniform tangential field (Nayyar & Murty 1960; Saville 1970, 1971*a*). Taylor demonstrated that some of the surface tension instabilities found by Rayleigh (1879) could be stabilized by a sufficiently large surface charge, while the growth rates of other increased. He also produced experimentally jets which exhibited remarkable longevity.

If charge relaxation can be considered instantaneous, Saville (1970) showed that an uncharged jet could be completely stabilized by a strong enough field. In a later paper, (Saville 1971 *a*), he showed that effects of finite charge relaxation time were destabilizing. The instability could be either oscillatory or direct, in contrast to the planar relaxation instabilities found by Melcher & Schwarz (1968) which are always oscillatory. We shall find in §7 that there is a field strength which optimizes stability in this case.

When the jet is produced in the cone–jet process, both tangential fields and surface charges are present. Furthermore, the effects of charge relaxation cannot be ignored. An electric current is observed to pass through the cone into the jet. In the conical region this current is usually carried by conduction, but when the jet is formed, and breaks up into drops, the current can only be carried by the fluid motion of charges (a *convection* current). Indeed, the width of the jet, and hence the size of droplets eventually produced, is determined partially by the requirement that a sufficiently large convection current be supported.

In this paper we investigate the interactions between surface charge, tangential field and charge relaxation with regard to the stability of the jet. A natural model to consider is that of a cylindrical jet of radius a with constant electrical permittivity ϵ and conductivity σ, carrying a uniform surface charge q_0 in a constant tangential electric field E_0. The exterior of the jet is assumed to be an insulator with vacuum permittivity ϵ_0. The electrical properties are then defined by the two parameters $\epsilon_0 E_0/q_0$ and ϵ/ϵ_0, together with some measure of field strength and the charge relaxation time. The linear stability may be investigated by perturbing the surface with modes $\propto e^{ikz+im\theta+st}$ in terms of cylindrical coordinates (r,θ,z) in a frame aligned and moving with the jet.

There are various interacting physical processes in the problem, each with its own characteristic timescale on which instabilities might be expected to grow. The timescales of charge relaxation, collapse due to surface tension and viscous diffusion are respectively

$$\tau_r \equiv \epsilon/\sigma, \quad \tau_c \equiv (\rho a^3/\gamma)^{1/2}, \quad \tau_v \equiv \rho a^2/\mu, \tag{1.1}$$

where γ, ρ and μ are the fluid's surface tension, density and viscosity. When the jet is produced by the cone–jet process, the fluid can choose its lengthscale a at will, but the other quantities are fixed for a given substance. If v_0 is a typical jet velocity, the ratio of conduction to convection currents is of the order $\pi a^2 \sigma E_0/2\pi a v_0 q_0 \approx a/\tau_r v_0$. For cone–jets this ratio is arguably $O(1)$ (Hines 1966). Typical values in MKS units based on Hayati *et al.* (1987) are $\mu = 2.5\times10^{-3}$ N s m^{-2}, $\gamma = 2.4\times10^{-2}$ N m^{-1}, $\sigma = 3.3\times10^{-7}$ ohm^{-1} m^{-1}, $\rho = 10^3$ kg m^{-3}, $\epsilon = 2\times10^{-11}$ F m^{-1} and $a = 2\times10^{-5}$ m, for which the above timescales in seconds are $\tau_r = 6\times10^{-5}$, $\tau_c = 1.8\times10^{-5}$ and $\tau_v = 1.6\times10^{-4}$. Viscous action is therefore comparatively slow and it makes sense to investigate the high-Reynolds-number limit, for which the growth rate s satisfies

$$|s| \gg \mu/\rho a^2 \quad \text{or} \quad |s\tau_v| \ll 1. \tag{1.2}$$

There is a difficulty associated with this limit. For simplicity we would like to assume an initially uniform jet velocity. However, a tangential stress, $q_0 E_0$, acts upon the fluid surface in the unperturbed state, and so the jet necessarily accelerates. We might argue that for sufficiently fast jets this acceleration can be neglected, just as we neglect the gravitational acceleration, g. However, the surface stress also generates vorticity which diffuses into the interior. If the viscosity is low, the shear can be large and in a fully diffused state the vertical velocity v_z is given by

$$v_z(r,t) = v_0 + \frac{q_0 E_0}{2\mu a}(r^2 - a^2) + \left(\frac{2q_0 E_0}{\rho a} + g\right)t, \tag{1.3}$$

assuming both gravity and the jet are aligned with the z-axis. If we wished to perturb about a genuinely steady state, we could require the tangential stress to balance gravity exactly so that the net acceleration of the jet is zero. More generally, the uniform accelaration may be neglected for vertical wavenumbers k such that

$$v_0^2 k \gg \left| \frac{2q_0 E_0}{\rho a} + g \right|. \tag{1.4}$$

However, the more serious assumption of uniform velocity and consequent neglect of vorticity in the stability analysis would be justified only if

$$\frac{\partial}{\partial t} \gg |\nabla u| \quad \text{or} \quad |s| \gg \frac{q_0 E_0}{\mu}. \tag{1.5}$$

In conjunction with (1.2), this would require $q_0 E_0 \ll \rho a^2 |s^2|$, which if electric effects are to be at all significant would imply that $\epsilon_0 E_0/q_0$ must either be very small or very large. Thus the assumptions of high Reynolds number and a uniform unperturbed jet are consistent only if either the surface charge or the tangential field is small. These limits are investigated in §§7 and 8. In the general case, we might anticipate instabilities on the shear timescale, τ_0,

$$\tau_0 \equiv s_0^{-1} \quad \text{where} \quad s_0 \equiv q_0 E_0/\mu, \tag{1.6}$$

as well as on the relaxation and capillary scales τ_r and τ_c and on some electric timescale. We shall refer to three electric timescales, τ_e, τ_q, τ_E and their inverses s_e, s_q, s_E, where

$$(s_e^2, s_q^2, s_E^2) \equiv \frac{(q_0 E_0, q_0^2/\epsilon_0, \epsilon_0 E_0^2)}{\rho a^2}. \tag{1.7}$$

As $s_0 = s_e^2 \tau_v$, we have $s_0 \gg s_e$ from (1.2), but if either q_0 or E_0 is small, s_E or s_q can be greater than s_0.

The main advantage of considering a uniform jet is that the perturbed flow remains irrotational apart from thin surface layers, which simplifies the analysis. However, provided the perturbation is axisymmetric ($m = 0$), similar simplification occurs for the quadratic r-variation of (1.3). This is because the potential vorticity, $(1/r)\partial v_z/\partial r$, is constant and the vortex lines are not disturbed by the perturbation. For this reason we shall concentrate on axisymmetric modes in §2 and §3, where we formulate the electrohydrodynamic stability problem. We nevertheless calculate the electric field perturbation for general m in §4 for use in §§7 and 8 and other limits. The cases $\tau_r \approx 0$ (high conductivity), $\tau_0 \approx 0$ (high shear), $q_0 \approx 0$ (low surface charge) and $E_0 \approx 0$ (low tangential field) are considered in §§5, 6, 7 and 8, and we conclude in §9.

2. Perturbation of the jet

We consider an incompressible cylindrical jet of radius a, uniform density ρ and viscosity μ, with $\nu = \mu/\rho$. We assume that gravity is negligible on the lengthscales of interest as in (1.4) and make a Galilean transformation to a frame moving with the surface of the jet. The velocity and pressure in the equilibrium state are given in terms of cylindrical coordinates (r, θ, z) by $\boldsymbol{u} = (0, 0, u_0(r)), p = p_0$, where from (1.3)

$$u_0 = -\tfrac{1}{2} a s_0 (1 - r^2/a^2). \tag{2.1}$$

In this accelerating frame we have Poiseuille flow back up the jet, driven by the fictitious inertial force. We consider a perturbation so that the surface S takes the form

$$r = a(1+\zeta) \quad \text{where} \quad \zeta = \delta\, \mathrm{e}^{\mathrm{i}kz + \mathrm{i}m\theta + st}, \tag{2.2}$$

with $\delta \ll 1$ and s being the (complex) growth rate of the disturbance. The wavenumbers k and m are real and positive but otherwise arbitrary save that m must be integral. For reasons discussed above, we shall take $m = 0$ unless $q_0 E_0$ is small. In terms of the non-dimensional wavenumber

$$\kappa = ka, \tag{2.3}$$

the unit normal to $S, \hat{\boldsymbol{n}}$, is given by

$$\hat{\boldsymbol{n}} = (1,0,0) - \mathrm{i}\zeta(0, m, \kappa) + O(\delta^2), \tag{2.4}$$

and the surface curvature, C, is given by

$$C = (\boldsymbol{\nabla} \cdot \hat{\boldsymbol{n}})|_S = \frac{1}{a} - \frac{\zeta}{a}(1 - m^2 - \kappa^2) + O(\delta^2). \tag{2.5}$$

Inside the perturbed jet, to leading order in δ the velocity $\boldsymbol{u}$ and pressure p take the form

$$\boldsymbol{u} = (0, 0, u_0(r)) + \zeta \boldsymbol{u}_1(r), \quad p = p_0 + \zeta p_1(r). \tag{2.6}$$

If we restrict ourselves to axisymmetric perturbations ($m = 0$), then the velocity is poloidal, while the vorticity remains azimuthal,

$$\boldsymbol{u}_1 = (u_r, 0, u_z), \quad \boldsymbol{\nabla} \wedge \boldsymbol{u} = \left(0, -\frac{r s_0}{a} + \zeta\omega(r), 0\right). \tag{2.7}$$

We may then define a streamfunction $\psi(r)$ such that

$$u_r = -\mathrm{i}k\psi/r, \quad u_z = \psi'/r \quad \text{and} \quad \omega r = -\mathrm{D}^2\psi, \tag{2.8}$$

where $'$ denotes differentiation with respect to the argument, and the Stokes operator

$$\mathrm{D}^2 \equiv \frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - k^2.$$

The vorticity equation takes the form

$$(s + \mathrm{i}ku_0)\, r\omega - u_r(ru_0'' - u_0') = \nu \mathrm{D}^2(\omega r) \tag{2.9}$$

(e.g. Drazin & Reid 1981, p. 217). For the parabolic profile (2.1) the middle term disappears from (2.9) and the solution may be found in terms of confluent hypergeometric functions (Pekeris 1948). Here we are interested in the low-viscosity limit, which it is easier to find directly from (2.9). In the absence of a source term, the vorticity perturbation ω is generated only at the boundary, from where it can diffuse but a short distance in the time available. The viscous terms and ω are thus negligible outside a surface boundary layer, and noting that $u_0(a) = 0$, we can write down the solution

$$\omega = -B\beta^2\, \mathrm{e}^{\beta(r-a)}(1 + O(\beta a)^{-1}), \tag{2.10}$$

for some constant B, where

$$\beta^2 = s/\nu \quad \text{with} \quad \mathrm{Re}\{\beta\} \geqslant 0 \quad \text{and} \quad |\beta a| \gg 1, \tag{2.11}$$

and we assume $k \ll \beta$. The streamfunction then takes the form

$$\psi = ArI_1(kr) + Ba\, \mathrm{e}^{\beta(r-a)}(1 + O(\beta a)^{-1}) \tag{2.12}$$

for some constant A, where I_1 is a modified Bessel function. The first term in (2.12) corresponds to irrotational flow. Standard boundary-layer arguments may be used to show that the pressure is constant across the surface layer, and so p_1 is determined by the z-component of the inviscid momentum equation evaluated outside the layer

$$(s+\mathrm{i}ku_0)\,u_z+u_r\,u_0' = -\mathrm{i}kp_1/\rho. \tag{2.13}$$

From (2.8) and (2.12), without the boundary-layer term and using the identity $I_1'(x)+I_1/x = I_0(x)$,

$$p_1 = -\rho A\left[\left(\frac{s}{\mathrm{i}k}+u_0(r)\right)kI_0(kr)-s_0\frac{r}{a}I_1(kr)\right]. \tag{2.14}$$

The constants A and B are determined by the perturbed surface stresses, T_n and T_t, which we write in the form

$$\left.\begin{aligned}|T_t| &= q_0\,E_0+\zeta T_z\\ T_n &= p_0+\zeta p_S\end{aligned}\right\}\quad\text{on}\quad r = a(1+\zeta). \tag{2.15}$$

The tangential viscous stress to leading order in βa is in the z-direction, so from (2.6)

$$|T_t| = \mu(0,0,1)\cdot\frac{\partial \boldsymbol{u}}{\partial r} = \mu(s_0(1+\zeta)+\zeta u_z'), \tag{2.16}$$

or from (2.8), (2.12), (2.15), (2.11) and (1.6),

$$B = \frac{T_z-q_0\,E_0}{\rho s}+O(\beta a)^{-1}. \tag{2.17}$$

The normal stress condition is

$$T_n = p_0+\zeta p_1+2\mu\hat{\boldsymbol{n}}\cdot[(\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}], \tag{2.18}$$

although the deviatoric stress could in fact be neglected in favour of the pressure for this high-Reynolds-number flow. There are some subtleties associated with this term, which we discuss in §3. We write

$$\hat{\boldsymbol{n}}\cdot[(\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}] = \zeta R, \tag{2.19}$$

where from (2.1), (2.4) and (2.6) $\mu R \sim q_0\,E_0$ as $\mu\to 0$. From (2.15) and (2.14),

$$p_S = \rho A[\mathrm{i}sI_0(\kappa)+s_0\,I_1(\kappa)]+2\mu R+O(\beta a)^{-1}. \tag{2.20}$$

Finally, we apply the kinematic boundary condition

$$\frac{\mathrm{D}}{\mathrm{D}t}(r-a\zeta) = 0\quad\text{on}\quad S, \tag{2.21}$$

where $\mathrm{D}/\mathrm{D}t$ is the material derivative, or, since $u_0(a) = 0$, to leading order in δ,

$$as = u_r(a) = -\mathrm{i}k(AI_1(\kappa)+B). \tag{2.22}$$

Combining (2.22), (2.20) and (2.17), we obtain the characteristic equation for the growth rate s

$$-\kappa(p_S-2\mu R) = \left(\frac{I_0}{I_1}-\mathrm{i}\frac{s_0}{s}\right)(\rho a^2s^2+T_z), \tag{2.23}$$

where
$$T_2 = i\kappa(T_z - q_0 E_0). \tag{2.24}$$

If p_S and T_z are independent of s, (2.23) is a cubic equation for s. Instability occurs if any root has a positive real part. In fact, we shall see in §5 that the perturbations to the electric surface stresses do depend on s if $\tau_r \neq 0$, but first we consider some simpler cases.

If no tangential stress acts on the equilibrium state, so that $q_0 E_0 = 0 = s_0$, the above analysis easily extends to non-axisymmetric modes. The appropriate relation is then

$$-\kappa p_S = \frac{I_m}{I'_m}(\rho a^2 s^2 + T_2), \tag{2.25}$$

where now
$$T_2 = i\kappa T_z + imT_\theta \tag{2.26}$$

and T_θ is the azimuthal stress perturbation. The Bessel function $I_m(\kappa)$ is such that $I_m > 0$ and $I'_m > 0$ for all m and $\kappa > 0$. In the absence of any tangential stresses, so that $T_2 = 0$, stability can therefore be inferred from (2.25) if the surface perturbation p_S is real and positive. The physical processes giving rise to p_S may then be interpreted as providing a simple restoring force. In the Plateau problem, a surface tension γ acts with a constant external pressure p_A. The appropriate boundary condition on S is $p = p_A + \gamma C$, whence from (2.5), (2.15) and (2.18) $p_0 = p_A + \gamma/a$ and

$$p_S = -\frac{\gamma}{a}(1 - m^2 - \kappa^2), \tag{2.27}$$

giving stability unless $m = 0$ and $\kappa < 1$. The most unstable mode is $\kappa \approx 0.7$ for which $s\tau_c \approx 0.3433$, where the capillary timescale τ_c is given by (1.1). These classical results were given by Rayleigh (1879), demonstrating the instability of jets to long-wavelength axisymmetric disturbances which causes them to break up into droplets. It is observed that the charged jets that occur in electrostatic spraying processes can exhibit greater stability than uncharged jets, although they too eventually break up. The main purpose of this paper is to understand this phenomenon.

Some insight is gained by considering a weak tangential stress of $O(\mu)$, and then letting $\mu \to 0$, so that $q_0 E_0 = 0 = T_2$, $m = 0$ but $s_0 \neq 0$. Equation (2.23) is then a quadratic in s which will have a root with a positive real part unless p_S is real and

$$\tfrac{1}{4}\rho a^2 s_0^2 \geqslant -\frac{I_0}{I_1}\kappa p_S. \tag{2.28}$$

This stability condition is weaker than $p_S \geqslant 0$, suggesting that the presence of vorticity in the basic flow may be stabilizing, at least for axisymmetric modes, as given by Leib & Goldstein (1986). In our problem, however, $T_2 \neq 0$ and in the next section we consider the influence of electric effects on the surface boundary condition.

3. Electro-quasi-statics

We suppose the liquid jet has constant electric conductivity σ and permittivity ϵ and is surrounded by an insulator of vacuum permittivity ϵ_0. The charge relaxation time τ_r and the relative permittivity λ are defined by

$$\tau_r = \epsilon/\sigma, \quad \text{and} \quad \lambda = \epsilon/\epsilon_0. \tag{3.1}$$

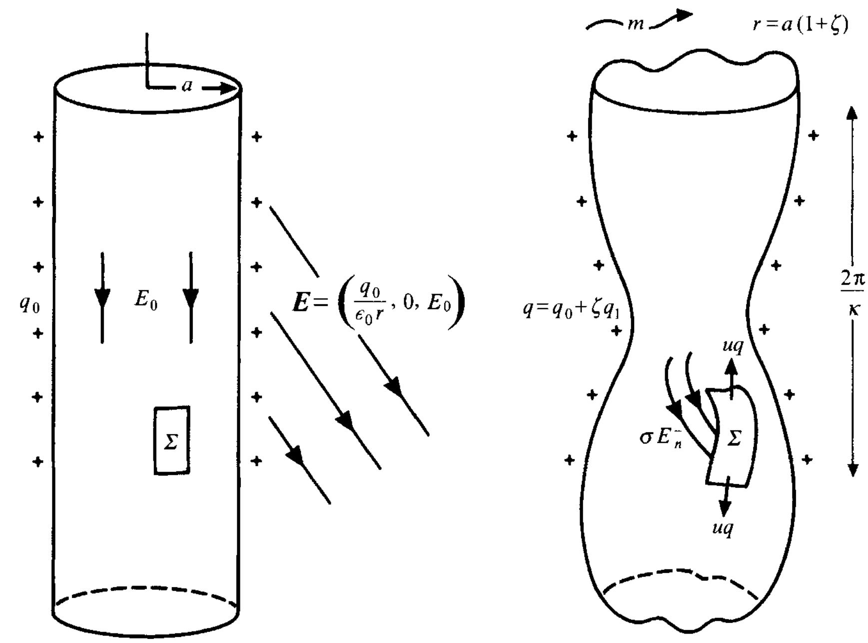

FIGURE 2. The perturbation of the electric field.

We denote by E^+ and E^- the electric field outside and inside the jet respectively, which we derive from potentials ϕ^+ and ϕ^- so that $E^\pm = -\nabla\phi^\pm$. These potentials satisfy Laplace's equation as there are no volume charges present.

The electrical properties are discontinuous across the surface S, giving rise to discontinuities in the electric field and a resultant surface stress. We denote by E_n and E_t the components of $E^\pm$ normal and tangential to the perturbed surface S. The jump conditions on S are

$$[E_t] \equiv E_t^+ - E_t^- = 0 \quad \text{or} \quad [\phi] = 0 \tag{3.2}$$

and

$$[\epsilon E_n] \equiv \epsilon_0 E_n^+ - \epsilon E_n^- = q_S, \tag{3.3}$$

where q_S is the surface charge density. One further relation reflecting the conservation of charge is required to determine the problem. Charge flows into the surface carried by the normal current, σE_n^-. Were the convection current due to fluid motions negligible in comparison, we would require

$$E_n^- = 0. \tag{3.4}$$

However, as discussed in the introduction, in this paper we are interested in processes for which the charge relaxation time and convection current are not negligible. We must therefore formulate the charge conservation law more precisely (Melcher & Schwarz 1968).

Consider an infinitesimal surface element Σ, moving with the fluid and supporting a variable surface charge density q_S as in figure 2. Equating the rate of change of charge over the element with the rate of charge arrival due to conduction current gives

$$\frac{\mathrm{D}}{\mathrm{D}t}(q_S \Sigma) = \sigma E_n^- \Sigma, \tag{3.5}$$

where $\mathrm{D}/\mathrm{D}t$ denotes the material derivative. Dividing through by Σ and using the formula for dilation of surface elements (e.g. Batchelor 1967, p. 132), we obtain

$$\sigma E_n^- = \frac{\mathrm{D}q_S}{\mathrm{D}t} - q_S \hat{\boldsymbol{n}} \cdot [(\hat{\boldsymbol{n}} \cdot \nabla) \boldsymbol{u}]. \tag{3.6}$$

Together with (3.3), (3.6) provides the appropriate relation between E_n^+ and E_n^- for general values of ϵ and σ. In the particular problem we are considering, the dilation factor is first order in δ as in (2.19), so that expanding the surface charge in the form

$$q_S = q_0 + \zeta q_1, \tag{3.7}$$

and from (2.19), noting that $u_0(a) = 0$, we obtain the linearized version of (3.6)

$$\sigma E_n^- = \zeta(sq_1 - q_0 R). \tag{3.8}$$

From (2.4) and (2.6), R is given by

$$\hat{\boldsymbol{n}} \cdot [(\hat{\boldsymbol{n}} \cdot \nabla) \boldsymbol{u}] \equiv \zeta R = \zeta(-\mathrm{i}\kappa u_0' + u_r') + O(\delta^2), \tag{3.9}$$

or from (2.8), (2.12) and (2.17)

$$R = \left[-Aik^2 I_1' - \frac{\beta T_2}{\rho a s} - \mathrm{i}\kappa s_0 \right] (1 + O(\beta a)^{-1}). \tag{3.10}$$

We note that as $\mu \to 0$ with T_2 fixed and $q_0 E_0 \neq 0$, the third term in (3.10) is larger than the second which is in turn larger than the first, by an order of $\mu^{-1/2}$ in each case. However, we shall see in §4 that T_2 depends on R and often the second and third terms are of equal magnitude. From (3.10) and (2.11),

$$R \approx -\mathrm{i}\kappa s_0 - \frac{T_2}{(\rho \mu a^2 s)^{1/2}}. \tag{3.11}$$

Equations (3.2) and (3.8) determine the electric field inside and outside the jet. The resultant stress on S may then be found from the Maxwell stress tensor, which is given in Cartesian form by

$$T_{ij} = \epsilon E_i E_j - \tfrac{1}{2} \epsilon |E|^2 \delta_{ij}. \tag{3.12}$$

If we denote by T_{nn} the jump in normal stress across the surface S, then

$$\begin{aligned} T_{nn} &= [\epsilon E_n^2] - \tfrac{1}{2}[\epsilon E_n^2] - \tfrac{1}{2}[\epsilon |E_t|^2] \\ &= \tfrac{1}{2}\epsilon_0 E_n^{+2} - \tfrac{1}{2}\epsilon E_n^{-2} + \tfrac{1}{2}(\epsilon - \epsilon_0) |E_t|^2, \end{aligned} \tag{3.13}$$

since $\boldsymbol{E}_t$ is continuous. The jump in tangential stress, $\boldsymbol{T}_{nt}$, takes the form

$$\boldsymbol{T}_{nt} = [\epsilon E_n \boldsymbol{E}_t] = \boldsymbol{E}_t [\epsilon E_n] = q_S \boldsymbol{E}_t, \tag{3.14}$$

from (3.2) and (3.3). If follows from (3.14) that a fluid supporting surface charge in a tangential field must be in motion, as the tangential electric stress can only be balanced by a viscous stress (Melcher & Taylor 1969). This basic electrohydrodynamic effect is responsible for the vorticity in our unperturbed flow and for the boundary-layer structure of the perturbation.

With electric effects included, the normal stress balance on S requires, from (2.18),

$$p + 2\mu R \zeta = p_A + \gamma C - T_{nn}, \tag{3.15}$$

which we write to lowest order in the form $p_0 = p_A + \gamma/a - T_0$. At order δ, the analogue of (2.27) is

$$p_S = -T_1 - \frac{\gamma}{a}(1 - m^2 - \kappa^2). \tag{3.16}$$

The quantity T_1 involves the first-order perturbation to T_{nn} which we calculate in the next section.

4. Perturbation of the electric field

We are considering a circular cylinder carrying a uniform surface charge q_0 in a constant tangential electric field $(0, 0, E_0)$. The surface is perturbed according to (2.2), giving rise to field perturbations proportional to ζ, where, in this section, m is not necessarily zero. The harmonic functions with the z and θ dependence of ζ involve once more the modified Bessel functions I_m and K_m, which must be chosen to guarantee regularity at $r = 0$ and $r = \infty$. The potentials $\phi^{\pm}$ therefore take the form

$$\left.\begin{aligned} \phi^+ &= -E_0 z - \frac{aq_0}{\epsilon_0}\log\frac{r}{a} + aC\zeta K_m(kr) \\ \phi^- &= -E_0 z + aD\zeta I_m(kr) \end{aligned}\right\} + O(\delta^2), \tag{4.1}$$

where C and D are constants to be determined from (3.2) and (3.8). Application of (3.2) on $r = a(1+\zeta)$ yields

$$-\frac{q_0}{\epsilon_0} + CK_m(\kappa) = DI_m(\kappa). \tag{4.2}$$

From (4.1) and (2.4) we find

$$E_n^+ = -\hat{\boldsymbol{n}} \cdot \nabla\phi^+ = \frac{q_0 a}{\epsilon_0 r} - \mathrm{i}\kappa\zeta E_0 - C\kappa\zeta K_m'(kr), \tag{4.3}$$

which on $r = a(1+\zeta)$ takes the form

$$E_n^+ = \frac{q_0}{\epsilon_0} - \zeta\left(\frac{q_0}{\epsilon_0} + \mathrm{i}\kappa E_0 + C\kappa K'\right). \tag{4.4}$$

In (4.4) and from now on, the suffix m and the argument κ will be neglected from the Bessel functions I and K. Similarly,

$$E_n^- = -\zeta(\mathrm{i}\kappa E_0 + D\kappa I'), \tag{4.5}$$

and so from (3.3) and (3.7)

$$q_1 = -\mathrm{i}\kappa E_0(\epsilon_0 - \epsilon) - \epsilon_0 C\kappa K' + \epsilon D\kappa I' - q_0. \tag{4.6}$$

Combining (3.8), (4.5) and (4.6), we obtain

$$D\alpha I' = CK' + \mathrm{i}E_0(1-\alpha) + \frac{q_0}{\epsilon_0 \kappa}\left(1 + \frac{R}{s}\right), \tag{4.7}$$

where

$$\alpha = \frac{\sigma}{\epsilon_0 s} + \frac{\epsilon}{\epsilon_0} = \left(\frac{1}{s\tau_r} + 1\right)\lambda. \tag{4.8}$$

The constants C and D may be found from (4.7) and (4.2), giving

$$\left.\begin{aligned} W_\alpha C &= \mathrm{i}E_0(1-\alpha)\,I+\frac{q_0}{\epsilon_0}\left(\frac{I}{\kappa}+\frac{IR}{s\kappa}+\alpha I'\right), \\ W_\alpha D &= \mathrm{i}E_0(1-\alpha)\,K+\frac{q_0}{\epsilon_0}\left(\frac{K}{\kappa}+\frac{KR}{s\kappa}+K'\right), \end{aligned}\right\} \tag{4.9}$$

where $\quad W_\alpha = \alpha KI'-IK'$ and for future use $W_\lambda = \lambda KI'-IK'$. (4.10)

We now calculate the perturbation to the normal surface stress. Since E_n^- is of order δ, we may rewrite (3.13) as

$$T_{nn} = \tfrac{1}{2}\epsilon_0\,E_n^{+2}+\tfrac{1}{2}(\epsilon-\epsilon_0)\,|E_t|^2+O(\delta^2). \tag{4.11}$$

The tangential field is most easily calculated from ϕ^-, giving

$$E_t = (0,0,E_0)+\mathrm{i}\zeta(\kappa E_0,\,-mDI,\,-\kappa DI)+O(\delta^2), \tag{4.12}$$

so that
$$|E_t|^2 = E_0^2-2\mathrm{i}\kappa E_0\,DI\zeta+O(\delta^2). \tag{4.13}$$

From (4.11), (4.4) and (4.13),

$$T_{nn} = \frac{q_0^2}{2\epsilon_0}-q_0\,\zeta\left(\frac{q_0}{\epsilon_0}+\mathrm{i}\kappa E_0+C\kappa K'\right)+\tfrac{1}{2}(\epsilon-\epsilon_0)\,E_0^2-\mathrm{i}\kappa(\epsilon-\epsilon_0)\,E_0\,DI\zeta. \tag{4.14}$$

Writing $T_{nn} = T_0+\zeta(T_1-2\mu R)$ as in (3.16) and using (4.9), we find

$$\begin{aligned} W_\alpha T_1 = &-\frac{q_0^2}{\epsilon_0}\left(\alpha KI'+\frac{K'IR}{s}+\kappa\alpha K'I'\right)-(\epsilon-\epsilon_0)\,E_0^2(\alpha-1)\,\kappa IK \\ &-\mathrm{i}q_0\,E_0\left[\alpha+(\lambda-1)\left(\kappa IK'+IK+\frac{IKR}{s}\right)\right]+2\mu RW_\alpha, \end{aligned} \tag{4.15}$$

where we have used the Wronskian identity $\kappa(KI'-IK') = 1$. Similarly, the tangential stress in (3.14) may be expanded $T_{nt} = (0,0,q_0\,E_0)+\zeta T_t'$, so that from (3.7), (3.14) and (4.12)

$$T_t' \equiv (T_r,T_\theta,T_z) = (\mathrm{i}\kappa E_0\,q_0,\,-\mathrm{i}q_0\,mDI,\,E_0\,q_1-\mathrm{i}\kappa DIq_0) \tag{4.16}$$

and hence, combining the definitions (2.24) and (2.26),

$$T_2 \equiv \mathrm{i}\kappa(T_z-q_0\,E_0)+\mathrm{i}mT_\theta = (\kappa^2+m^2)\,q_0\,DI+\mathrm{i}\kappa E_0(q_1-q_0), \tag{4.17}$$

which can be found from (4.6) and (4.9). We are now in a position to write down the dispersion relation for axisymmetric modes, given by (2.23), (3.16), (4.15) and (4.17), namely

$$\left(\frac{I_0}{I_1}-\mathrm{i}\frac{s_0}{s}\right)(\rho a^2s^2+T_2) = \frac{\gamma}{a}\kappa(1-\kappa^2)+\kappa T_1. \tag{4.18}$$

This would be a cubic equation in s were it not for the s dependence of T_1 and T_2. The condition for stability is that each root should have a negative real part. In addition to the wavenumber κ, equation (4.18) depends on five parameters: λ and four suitable ratios of the timescales τ_r, τ_c, τ_E, τ_q and τ_0. For given values of these parameters, the most unstable wavenumber and corresponding growth rate may be readily found, but it is difficult to discuss the behaviour in general without further simplification. We have already taken the low-viscosity limit $\tau_v \to 0$ and we now proceed to examine various further asymptotic limits. In §6 we consider the high-vorticity limit $\tau_0 \to 0$, while in §§7 and 8 we discuss the cases of low surface charge $\tau_q \to \infty$ and low tangential field $\tau_E \to \infty$. First, we investigate the high-conductivity limit $\tau_r \to 0$.

5. The high-conductivity limit

When the charge relaxation time is small, the field responds instantaneously to the surface perturbation, with the result that the surface stresses are independent of s. As we let $\tau_r \to 0$ and $\alpha \to \infty$, equations (4.6), (4.15) and (4.17) become

$$q_1 = -q_0\left(1+\frac{\kappa K'}{K}\right)-\frac{\mathrm{i}\epsilon_0 E_0}{KI'}, \tag{5.1}$$

$$T_1 = -\frac{q_0^2}{\epsilon_0}\left(1+\frac{\kappa K'}{K}\right)-(\epsilon-\epsilon_0)E_0^2\frac{\kappa I}{I'}-\frac{\mathrm{i}q_0 E_0}{KI'}+2\mu R \tag{5.2}$$

and
$$T_2 = -\mathrm{i}q_0 E_0\left[2\kappa+(\kappa^2+m^2)\frac{I}{I'}+\kappa^2\frac{K'}{K}\right]+\frac{\kappa}{KI'}\epsilon_0 E_0^2. \tag{5.3}$$

The phase difference between effects due to the surface charge q_0 and the tangential field E_0 is clear from the appearance of i in (5.1). This gives rise to oscillatory electrical instabilities in many limits. We begin by considering some special cases.

When $E_0 = 0 = R$, we have $T_2 = 0$ while T_1 is real and proportional to $(1+\kappa K'/K)$. This expression is positive only for $m = 0$ and $0 < \kappa < 0.6$, approximately. From (4.18) with $s_0 = 0$ we see that electric effects help to stabilize the jet against the axisymmetric Rayleigh instability for small wavenumbers. However, for $0.6 < \kappa < 1$, a range which includes the most unstable mode, both electrostatic and surface tension forces are destabilizing. It therefore appears that surface charge should enhance the instability of a jet. This result was given by Taylor (1969) correcting a flawed equation of Bassett (1894).

However, adding a small tangential field and a consequent $O(1)$ shear to the basic flow can lead to the opposite conclusion. If $E_0 = O(\mu)$, then we may take consistently $s_0 = O(1)$ but $E_0 \to 0$. As we saw in §2, the stability condition may then be written

$$\tfrac{1}{4}s_0^2 \geqslant \frac{\kappa I}{I'}\left[s_c^2(1-\kappa^2)-s_q^2\left(1+\frac{\kappa K'}{K}\right)\right], \tag{5.4}$$

where $s_c = \tau_c^{-1}$ as in (1.1). A large enough value of s_0 will stabilize all axisymmetric modes in this limit.

When $q_0 = 0$, both T_1 and $-T_2$ are real and negative for all m and κ since $\epsilon \geqslant \epsilon_0$ for physical reasons. Equation (4.18) indicates that electric effects are stabilizing for all wavenumbers in this limit, as given by Saville (1970). Introduction of small q_0 and consequent $O(1)$ values of s_0 can once again be shown to have a stabilizing influence.

We suppose now that $q_0/\epsilon_0 E_0$ is $O(1)$. As $q_0 E_0 \neq 0$, we take $m = 0$ and rewrite (4.18) in the form

$$\frac{I}{I'}s^3-\mathrm{i}s_0 s^2+\frac{s}{\rho a^2}\left[\frac{I}{I'}T_2-\frac{\gamma}{a}\kappa(1-\kappa^2)-\kappa T_1\right]-\mathrm{i}s_0\frac{T_2}{\rho a^2} = 0. \tag{5.5}$$

A typical value of T_1 and T_2 is $q_0 E_0 \equiv \mu s_0$. As the viscosity is small, from (1.2) we must have

$$\rho a^2 s_0|s| \gg |T_1|, |T_2|, \tag{5.6}$$

and so (5.5) is to the same order of approximation

$$\frac{I}{I'}s^3 - \mathrm{i}s_0 s^2 - s_c^2 \kappa(1-\kappa^2)s - \mathrm{i}s_0 \frac{T_2}{\rho a^2} = 0. \tag{5.7}$$

The sum of the roots of (5.7) is purely imaginary, but their product is not, as T_2 is strictly complex. It follows that at least one root has a positive real part. However, we are only concerned with timescales shorter than the viscous scale τ_v, and it may be that the instability is slow. We must distinguish between the cases $\tau_c \sim \tau_0$ and $\tau_c \gg \tau_0$. (If $\tau_c \ll \tau_0$ then none of the electric effects can compete with the Rayleigh surface tension instability.) If surface tension is no larger than the electric stresses, then $\tau_c \gg \tau_0$ and by an argument similar to the above we may set $s_c = 0$ in (5.7). One of the roots of (5.7) is then rapidly oscillating on the shear timescale τ_0,

$$s = \mathrm{i}s_0 \frac{I'}{I} + O(\tau_v^{-1}). \tag{5.8}$$

This root corresponds to neutrally stable inertial oscillations. In fact the real part of (5.8) is negative, but this is not relevant as we have already neglected terms of this magnitude. The other two roots are given to leading order by

$$s^2 = -\frac{T_2}{\rho a^2} = s_E^2\left[\frac{-\kappa}{KI'} + \mathrm{i}\frac{q_0}{\epsilon_0 E_0}\left(2\kappa + \frac{\kappa^2 I}{I'} + \frac{\kappa^2 K'}{K}\right)\right]. \tag{5.9}$$

One of these latter two will be unstable on an electric timescale. We find that the real part of this unstable root is a monotonic increasing function of κ, and so the largest growth rate may be found by letting $\kappa \to \infty$, when

$$s^2 \sim s_E^2\left[-2\kappa^2 + 2\mathrm{i}\kappa\frac{q_0}{\epsilon_0 E_0}\right], \tag{5.10}$$

using the asymptotic form of the Bessel functions, or

$$s \sim \sqrt{2}\,\mathrm{i}\kappa s_E + s_q/\sqrt{2}. \tag{5.11}$$

Once again we have a rapid oscillation, but one which grows on the electric timescale τ_q. The neglected surface tension term in (5.7), which becomes significant when

$$\kappa^2 \sim s_0 s_E / s_c^2, \tag{5.12}$$

limits the size of the most unstable wavenumber.

If the capillary time is short, so that $\tau_c \sim \tau_0$, then two of the roots of (5.7) are large, satisfying

$$\frac{I}{I'}s^2 - \mathrm{i}s_0 s - \kappa(1-\kappa^2)s_c^2 = O(s_e^2). \tag{5.13}$$

This case we considered at the end of §2 and in (5.4). On the fast timescale we have stability provided

$$s_0^2\tau_c^2 \geq 4\kappa(1-\kappa^2)I/I'. \tag{5.14}$$

The long waves ($\kappa \to 0$) are the last to be stabilized, requiring $s_0 \geq \sqrt{2}\,s_c$. The other root of (5.7) is small, $s \sim s_e^2/s_0 = \tau_v^{-1}$. Indeed, the roots of (5.13) also have real parts of this order, but these must be neglected for consistency with our other approximations.

We conclude that for highly conducting jets, the axisymmetric modes can be stabilized by a suitable shear. This is consistent with the experiments of Huebner (1969) which showed that non-axisymmetric modes could cause the break-up of water jets. If the shear is too great, however, then axisymmetric oscillatory instabilities are to be expected on the electric timescale τ_q.

6. High shear

In this section we consider the case where the shear timescale is shorter than those of charge relaxation or surface tension. In some ways this limit is a natural extension of the assumption of low viscosity. As we let $s_0 \to \infty$ in (4.18), we see that there will be one root with $s \sim s_0 \gg s_c$,

$$s = \mathrm{i}s_0 I'/I + O(s_c^2/s_0) + O(\tau_v^{-1}), \tag{6.1}$$

corresponding to neutrally stable inertial oscillations which do not interest us. The other roots have $s \ll s_0$ for which the term in the first bracket in (4.18) is large, and as $T_2 \sim T_1$, we must require

$$s^2 + \frac{T_2}{\rho a^2} \approx \frac{\mathrm{i}s}{s_0} s_c^2 \kappa(1-\kappa^2). \tag{6.2}$$

To begin with, we neglect the capillary term on the right-hand side of (6.2) as $\mu \to 0$, assuming $|s| \gg \kappa^3 s_c^2/s_0$. In fact we shall find that the most unstable modes have a wavenumber large enough for this term to be significant. Unlike in the case of rapid charge relaxation, the electric stresses may need time to adjust to the perturbation, so that T_2 depends on s. Furthermore, both T_2 and the surface dilation factor R are large. Since $R \sim s_0$ and $s_0 \tau_r \gg 1$ in this limit, not only is $R/s \gg 1$, but also $R/s \gg \alpha$, from (4.8). We repeat (3.11),

$$R = -\mathrm{i}\kappa s_0 - \frac{T_2}{q_0 E_0}\left(\frac{s_0}{s}\right)^{1/2} s_e, \tag{6.3}$$

and calculate T_2 from (4.17) in the limit of large (R/s):

$$\frac{T_2}{q_0 E_0} = \frac{\kappa R}{s W_\alpha}\left[\frac{q_0}{\epsilon_0 E_0} IK + \mathrm{i} W_\lambda\right]. \tag{6.4}$$

We see from (4.8) and (4.10) that W_α depends on $s\tau_r$. The size of this product depends on the order of $s_0 \tau_r^3/\tau_e^2$, which could be small even though $s_0 \tau_r \gg 1$. If $s\tau_r \ll 1$, then $W_\alpha = \lambda K I'/s\tau_r$ and from (6.2), (6.3) and (6.4)

$$s^2 = \frac{s_0 \tau_r}{\tau_e^2}\left[\frac{q_0}{\epsilon_0 E_0}\frac{\mathrm{i}\kappa^2 I}{\lambda I'} - \frac{\kappa^2 W_\lambda}{\lambda K I'}\right]. \tag{6.5}$$

The growth rate $\mathrm{Re}[s]$ is an increasing function of q_0 and κ, the latter limited by surface tension. Rapidly growing short waves will appear, with $s \sim \kappa s_q (s_0 \tau_r)^{1/2}$.

If $s\tau_r \gg 1$, then $W_\alpha \approx W_\lambda$ from (4.10). Combining (6.2), (6.3) and (6.4), and writing

$$n = \frac{1}{\kappa}\left(\frac{s^3 \tau_e^2}{s_0}\right)^{1/2}, \tag{6.6}$$

we obtain $$n^2 + bn = \mathrm{i}b, \quad \text{where} \quad b = \left(\mathrm{i} + \frac{q_0}{\epsilon_0 E_0}\frac{IK}{\kappa W_\lambda}\right). \tag{6.7}$$

These last two equations define six values of s. However, defining arguments to lie in the range $(-\pi, \pi)$, we must require from (2.11) that $\pi/2 \geq |\arg(s^{1/2})|$, which halves the number of roots. Stability would require all roots in the range $\pi/2 \geqslant |\arg(s^{1/2})| \geqslant \pi/4$, so that $\pi \geqslant |\arg(n)| \geqslant 3\pi/4$. It turns out that there is always at least one unstable root. We conclude that as $\mu \rightarrow 0$ we will encounter here also rapidly growing ($s \sim [\kappa^2/\mu]^{1/3}$) short waves, of a size limited by surface tension. It is interesting to note that the terms in (6.3) are of the same order. The relative slowness of charge relaxation leads to large tangential stress perturbations, which in turn give rise to large tangential velocities in an effort to redistribute the surface charge.

We now consider the effect of the surface tension term in (6.2), which is negligible as $s_0 \rightarrow \infty$ for fixed κ unless s_c is also large. As s_c increases, the terms in (6.2) become of the same order when $s_0^2 s_e \sim s_c^3$ with the growth rate still of the order $s \sim (s_0 s_e^2)^{1/3}$. If s_c increases further the electrical terms are small and to leading order we recover the stability condition (5.14). Assuming this is satisfied, the growth rate will be small, of the order

$$\mathrm{Re}[s] \sim s_e^2 s_0^3/s_c^4 = \tau_v^{-1}(s_0/s_c)^4. \tag{6.8}$$

When $s_c \sim s_0$ we recover the familiar possibility of stability on timescales less than τ_v. We conclude that jets for which the shear is much larger than surface tension will be highly unstable, but a suitable balance between the two is stabilizing, at least for axisymmetric modes.

7. Low surface charge

The analysis simplifies greatly when the tangential field E_0 dominates the surface charge q_0, so that

$$q_0/(\epsilon_0 E_0) \ll 1. \tag{7.1}$$

In this limit it is quite consistent to neglect the tangential stress on, and non-uniform velocity of the unperturbed jet at high Reynolds number. We shall therefore include non-zero m in the discussion, as outlined in §2. As we let $q_0 \rightarrow 0$ in §4, we find

$$\left.\begin{aligned} W_\alpha T_1 &= -(\lambda-1) IK[\epsilon_0 E_0^2 \kappa(\alpha-1) + \mathrm{i} q_0 E_0 R/s], \\ \text{and}\qquad W_\alpha T_2 &= \epsilon_0 E_0^2 \frac{\kappa}{s\tau_r}\lambda + \mathrm{i} q_0 E_0 \kappa W_\lambda \frac{R}{s}. \end{aligned}\right\} \tag{7.2}$$

As we observed in §3, the surface dilation factor R is potentially large as $\mu \rightarrow 0$, and so we include in (7.2) terms involving the product $q_0 R$. In fact, provided

$$s \gtrsim s_0 \equiv q_0 E_0/\mu, \tag{7.3}$$

the effects of small q_0 may be considered as a regular perturbation of the case $q_0 = 0$, which we examine first. The low-conductivity limit was considered by Nayyar & Murty (1960). If take the high-conductivity limit, $\tau_r \rightarrow 0$ and $\alpha \rightarrow \infty$, we obtain from (4.18) and (7.2)

$$s^2 \frac{I}{\kappa I'} = s_c^2(1-m^2-\kappa^2) - s_E^2\left[\frac{\kappa I}{I'} + \frac{I}{K I'^2}\right], \tag{7.4}$$

as given by Saville (1970). Both of the electric terms in (7.4) are stabilizing for all wavenumbers. A sufficiently high value of E_0, or more precisely of $a\epsilon_0 E_0^2/\gamma$, will therefore guarantee stability of the jet in this limit, as discussed in §5. Stability for given s_E and $m = 0$ will be afforded also by a suitable shear s_0.

When $q_0 = 0$ but τ_r is not negligible, the characteristic equation (4.18) is a cubic in s, which we write as

$$\hat{s}^3 + a_2 \hat{s}^2 + a_1 \hat{s} + a_0 = 0, \tag{7.5}$$

where $\hat{s} = s\tau_r$, and a_0, a_1 and a_2 are given below. The stability condition is that all three roots should have a negative real part. In general, the number of roots of a polynomial in a half-plane may be found using the principle of the argument (Henrici 1974, p. 485). Here, the coefficients a_i are real, and the appropriate conditions for strict stability are

$$a_2 > 0, \quad a_0 > 0 \quad \text{and} \quad a_2 a_1 > a_0. \tag{7.6}$$

We introduce the non-dimensional parameters E and G by

$$E^2 = \frac{\epsilon_0 E_0^2 \tau_r^2}{\rho a^2} = \frac{\tau_r^2}{\tau_E^2} \quad \text{and} \quad G = \frac{\gamma \tau_r^2}{\rho a^3} = \frac{\tau_r^2}{\tau_c^2}. \tag{7.7}$$

Then (7.5) is equivalent to (4.18) provided

$$\left.\begin{aligned} a_2 &= \lambda K I'/W_\lambda, \\ a_1 &= \frac{(\lambda-1)^2 E^2 \kappa^2 I' K}{W_\lambda} - G(1-m^2-\kappa^2)\frac{\kappa I'}{I}, \\ a_0 &= \lambda \kappa^2 E^2 - G(1-m^2-\kappa^2)\frac{\kappa \lambda K I'^2}{W_\lambda I}, \end{aligned}\right\} \tag{7.8}$$

as given by Saville (1971*a*) with different notation. Now since $\lambda \geqslant 1$, $KI' > 0$ and $IK' < 0$, it follows from (4.10) that $W_\lambda > 0$ and that $1 > a_2 > 0$. The first condition in (7.6), which is equivalent to the sum of the roots of (7.5) being negative, is thus satisfied for all m and κ. After some manipulation, the condition $a_0 > 0$ is seen to be precisely the same stability condition as occurs in the high-conductivity limit, namely that the right-hand side of (7.4) should be negative. However,

$$a_2 a_1 - a_0 = -\frac{E^2 \kappa \lambda}{W_\lambda^2}[(\lambda-1)KI' + W_\lambda] < 0. \tag{7.9}$$

The final requirement for stability in (7.6) is thus violated for all wavenumbers. As surface tension does not appear in (7.9), it is tempting to regard the instability as a wholly electrical effect, due to charge relaxation. However, the unstable roots of (7.5) may be either real or complex, and it is natural to identify the real instabilities as being driven predominantly by surface tension, and the complex ones by charge over-relaxation. It is interesting that the interaction of the two processes can produce enhanced stability behaviour. For fixed values of κ and the other parameters, as E is increased from zero, the cubic (7.5) has two real positive roots which move together, with the maximum growth rate diminishing. These coalesce and form a conjugate pair whose real part then increases. When κ is allowed to vary the picture is broadly similar, and for given values of λ and G there is a value of E which gives optimal stability.

Some of the growth rates are given by Saville (1971*a*) for the case $\lambda = 78$ appropriate to water, which we reproduce below. As we discussed in the introduction, there are some difficulties in producing aqueous jets via the cone–jet process, but these can be overcome. We are also interested here in lower values of λ appropriate to Hayati *et al.*'s (1987) experiments.

In the absence of any electric field, the most unstable Rayleigh mode is $m = 0$, $\kappa \approx 0.7$, for which the growth rate $s_T \approx 0.3433 s_c$. As the electric field is increased for fixed G and λ, the maximum growth rate, s_M, decreases until some minimum value, s_{min}, is attained when $E = E_{min}$. At this critical value of E two different wavenumbers have the same maximal growth rate. As E is increased through E_{min} the wavenumber of the most unstable mode, κ_M, is discontinuous, as is the derivative of s_M with respect to E.

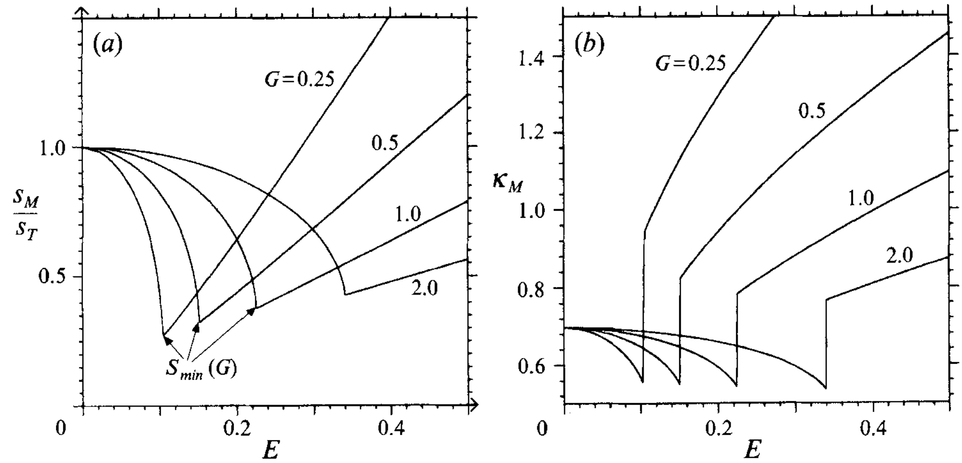

FIGURE 3. (*a*) The maximum growth rate, s_M, and (*b*) the corresponding wavenumber κ_M, as functions of E for $q_0 = 0$, $\lambda = 2$ and $G = 0.25, 0.5, 1, 2$.

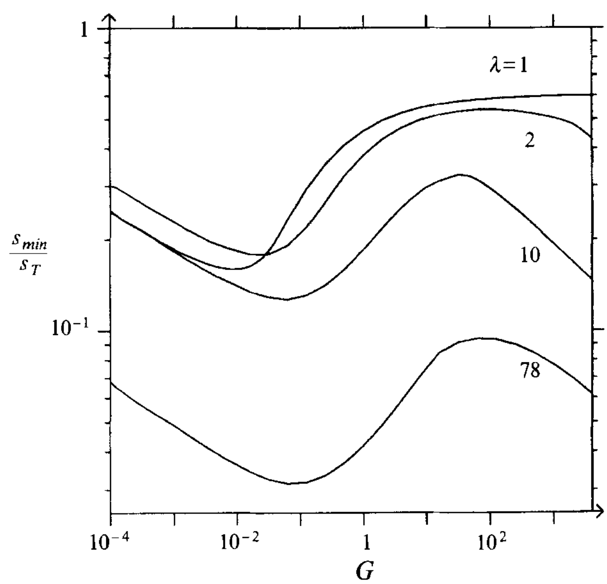

FIGURE 4. For an uncharged jet, the smallest value as E varies of the maximal growth rate, s_{min}/s_T, as a function of G for $\lambda = 1, 2, 10, 78$.

This behaviour is illustrated in figure 3 for the case $\lambda = 2$ and various values of G. The degree to which stability can be improved by a suitably chosen field strength is determined by s_{min}. Typically a jet of speed v_0 can travel a distance of order v_0/s_{min} before breaking up. The longevity of the jet can therefore be increased by a suitable value of E, even when $q_0 = 0 = s_0$. In figure 4, the minimum value of the maximum growth rate, s_{min}, is drawn on a log–log graph against G, for various values of λ; s_{min} is expressed as a proportion of s_T, the largest growth rate when $E = 0$. It is found that the axisymmetric mode $m = 0$ is always the most unstable, although the difference is not great for those modes with a large oscillatory component. As $G \to 0$ we find that $E_{min} \propto G$, the most unstable wavenumber increases and the instability becomes highly oscillatory, as given by Saville. As G increases, E_{min} increases monotonically and faster than linearly.

We can now consider the effect of introducing a small surface charge q_0. This first manifests itself by means of the shear rate s_0 in (4.18). The multiplicative factor of i in (4.18) enables the effects of small s_0 to be understood easily. If $\hat{s} = s^*$ is a real root of (7.5), then it will be perturbed by the introduction of s_0 in (4.18) to a root of the form $\hat{s} = s^* + s_0 \tau_r s_1$, where s_1 is purely imaginary. To leading order, the effect on the growth rate is to introduce a small oscillatory component. If s^* is either of a pair of complex-conjugate roots, then s_1 will be i times one of two complex conjugates. The real part of one of the complex roots will therefore be increased by the perturbation, while the other will be decreased. The net effect is to increase the growth rate of instabilities. As q_0 is increased further, s_0 becomes $O(s_E)$ before the R/s terms in (7.2) become significant. We then have a quartic equation for $\hat{s}$,

$$\hat{s}^4 + (a_2 - \mathrm{i}d)\hat{s}^3 + (a_1 - \mathrm{i}da_2)\hat{s}^2 + a_0\hat{s} - \mathrm{i}d\lambda\kappa E^2/W_\lambda = 0, \tag{7.10}$$

where $d = s_0 \tau_r I'/I$. We have seen that when $s_c \gg s_E$, a sufficiently large shear s_0 can suppress the capillary instability, and we might wonder whether it could also control charge over-relaxation. However, it turns out that a necessary condition for all the roots of (7.10) to lie in the left-half plane is once again $a_0 < a_1 a_2$ which we know to be universally false. All modes are unstable on the charge relaxation time τ_r, even in the presence of shear.

8. Low tangential field

Another limit affording great simplification is that when the component of electric field along the jet, E_0, is negligible so that

$$q_0/(\epsilon_0 E_0) \gg 1. \tag{8.1}$$

Once again, the shear s_0 is small in this limit, and so we may self-consistently consider $m \neq 0$. In contrast to §7, the presence of surface charge in the unperturbed state renders the jet sensitive to changes in surface area, especially at low conductivity. If we set $E_0 = 0 = s_0$ in (3.11) we see that either $T_2 \to 0$ or $R \sim T_2/\mu^{1/2}$ as $\mu \to 0$. This latter alternative is inconsistent with (4.17) from which we find that $T_2 \sim R$. It is therefore necessary for the dilation factor R to be such that $T_2 \approx 0$ in this limit. This requires

$$R/s \approx -(1 + \kappa K'/K) \tag{8.2}$$

and hence from (3.11)

$$T_2 = O[(\rho\mu a^2 s^3)^{1/2}]. \tag{8.3}$$

Using (4.15) and (8.2), to leading order in μ and E_0, equation (4.18) becomes

$$\frac{I}{I'}s^2 = s_c^2\kappa(1 - m^2 - \kappa^2) - s_q^2\kappa\left(1 + \frac{\kappa K'}{K}\right) + \mathrm{i}s_0 s - \frac{I}{I'}\frac{T_2}{\rho a^2}. \tag{8.4}$$

The last two terms in (8.4) must be regarded as small, initially, although we can allow $s_0 \sim s_q$ if we take $m = 0$. When $E_0 = 0$ and $\mu \to 0$, a little surprisingly we obtain the same relation as in the high-conductivity limit, obtained in a flawed form by Bassett (1894) and corrected by Taylor (1969). This is plausible on physical grounds. The large surface velocities induced at high Reynolds number by a tangential stress have the same effect as a small charge relaxation time in redistributing surface charge instantly. In each case the internal field E^- remains zero and so the permittivity ratio λ is irrelevant. As we discussed in §5, when $E_0 = 0$ modes with $m = 0$ and $0.6 < \kappa < 1$ are always unstable, while others can be stabilized for suitable choice of q_0. Since the most unstable Rayleigh mode has $\kappa \approx 0.7$ it is perhaps not surprising that s_M, the maximum growth

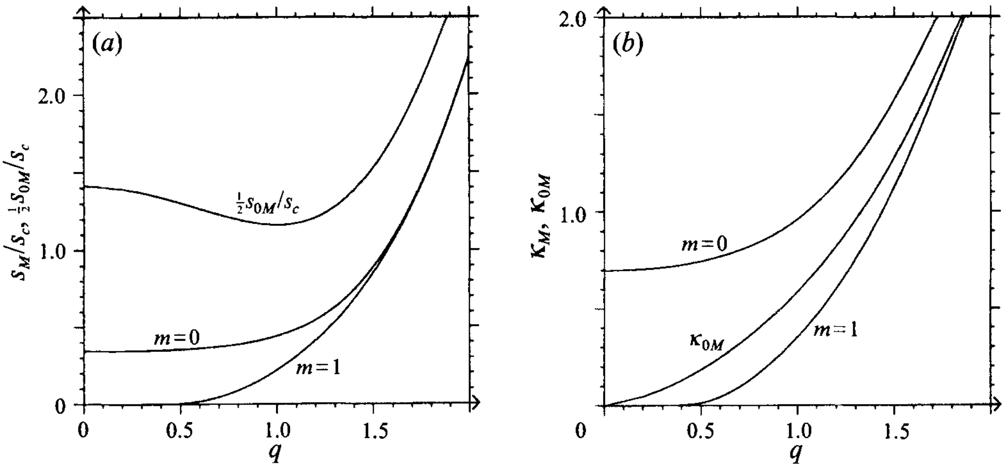

FIGURE 5. (a) The maximal growth rate s_M/s_c, and (b) most unstable mode κ_M, as functions of q for $E_0 = 0 = s_0$ and $m = 0$, 1. λ is arbitrary. Half the smallest value of s_0 needed to stabilize all axisymmetric modes ($\frac{1}{2}s_{0M}/s_c$) and the last mode to be stabilized (κ_{0M}) are also shown.

rate taken over all κ and m, is an increasing function of q_0. In figure 5, s_M/s_c and the corresponding wavenumber κ_M are plotted against $q \equiv s_q/s_c = (q_0^2 a/\epsilon_0 \gamma)^{1/2}$ for $m = 0$ and 1.

Once again, the effect of introducing small E_0 as a perturbation manifests itself first by means of s_0 As we have seen, a suitable value of s_0 can stabilize all axisymmetric modes, and the least such value satisfies (5.4) with equality. We illustrate this minimum value $s_0 = s_{0M}(q)$ occurring when $\kappa = \kappa_{0M}(q)$, in figure 5.

It is plausible, but not certain, that the limit of this section is the most appropriate for many jets occurring in the cone–jet process. If, as is usually the case, the drops formed by the eventual break-up of the jet are charged close to their Rayleigh limit, then $s_q \sim s_c$. Optimal stability is then afforded by a weak tangential field E_0 such that $s_0 \sim s_c \sim s_q$ or

$$E_0 \sim \frac{\mu}{a(\rho\epsilon_0)^{1/2}}. \tag{8.5}$$

In practice smaller tangential fields are encountered, although it is not always clear what value of E_0 is appropriate for a given experiment.

9. Concluding remarks

In this paper we have examined the stability implications at high Reynolds number of the interaction between surface charge, tangential field and finite charge relaxation rates, all of which are present in practical devices. In the absence of electric effects, surface tension renders the cylindrical jet unstable to axisymmetric modes. Loosely speaking, the addition of surface charge alone accentuates the instability. A slightly surprising feature exhibited by poor conductors is that the small fluid viscosity performs to some extent the role of a high conductivity in permitting the rapid redeployment of surface charge.

In contrast, the addition of a tangential field alone can suppress the capillary instability when the conductivity is high. For moderate conductors, however, charge relaxation instabilities appear, which tend to be oscillatory rather than direct. The effects due to the appearance of surface charge are 90° out of phase with those due to tangential field and the interaction between the two guarantees that all modes are unstable. Nevertheless, a suitable value of the field reduces the overall growth rate of instabilities.

When both q_0 and E_0 are present, the primary effect is to introduce shear into the unperturbed flow. Assuming that this shear is fully diffused, it can readily control the axisymmetric capillary instability, with the effect of increasing significantly the lifetime of the jet. This is especially the case for relatively weak fields which, at high Reynolds number, can support shear within the jet without exciting electrical instabilities unduly. Higher field values give rise to rapid oscillatory instability whose wavelength is small, limited only by surface tension. Greatest stability occurs for jets with $s_c \sim s_0 \gg s_e$, when the disturbances grow only on the viscous timescale.

When either q_0 or E_0 vanishes, axisymmetric modes appear to be the most unstable at high Reynolds number, although not by very much when the growth rate s has a large imaginary part. This suggests that the restriction of this work to axisymmetric modes when $q_0/\epsilon_0 E_0$ is $O(1)$ may not be too serious a limitation. When the axisymmetric modes are all but stabilized by a suitable shear, however, it may well be that the $m = 1$ mode is the most unstable, as suggested by Huebner's (1969) experiments. Moreover, at low Reynolds numbers, with $E_0 = 0$ and $\tau_r = 0$, Saville (1971*b*) found that the most unstable modes could have $m = 1$, in keeping with Taylor's (1969) observations of very viscous jets. The high-viscosity limit warrants a more general investigation.

Perhaps the weak point of the model discussed in this paper is the assumption of a fully diffused state. We assume that viscous diffusion times are long, but demand that a suitable equilibrium be set up before we allow a surface disturbance to appear! The justification for such a model relies on the origin of the jet. The motivation for this work lies in the cone–jet process depicted in figure 1. A fully self-consistent model of this phenomenon has yet to be presented, but in high-Reynolds-number regimes, with a recirculating core flow as in Hayati *et al.* (1987), the jet appears to originate from a surface boundary layer, and is presumably well laden with vorticity at its formation (Mestel 1994). The fully diffused model studied here is arguably appropriate for such a situation. The alternative would be to examine a uniform jet with a growing surface layer as the unperturbed state, perhaps using spatial rather than temporal modes. There are arguments for using spatial modes for jet stability analysis, but such an approach is complicated enough even in the absence of electrical effects (Leib & Goldstein 1986), and has not been considered.

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